Universal Self Force from an Extended Object Approach

Amos Ori and Eran Rosenthal
Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel
(February 7, 2008)

We present a consistent extended-object approach for determining the self force acting on an accelerating charged particle. In this approach one considers an extended charged object of finite size ϵ, and calculates the overall contribution of the mutual electromagnetic forces. Previous implementations of this approach yielded divergent terms $\propto 1/\epsilon$ that could not be cured by mass-renormalization. Here we explain the origin of this problem and fix it. We obtain a consistent, universal, expression for the extended-object self force, which conforms with Dirac's well known formula.

When a charged particle is accelerated in a non-uniform manner, it exerts a force on itself. This phenomenon of *self force* (often called "radiation-reaction force") is known for almost a century, since the pioneer works by Abraham [1,2] and Lorentz [3] on the structure of the electron. The non-relativistic form of this force was obtained by Abraham and Lorentz, who found it to be proportional to the time-derivative of the acceleration. Later Dirac [4] derived the covariant relativistic expression for the self force acting on a point-like particle [Eq. (8) below].

The self-force is a remarkable phenomenon, because essentially it means that a charged particle may "exert a force on itself". A natural approach for comprehending this phenomenon within the framework of classical electrodynamics is the *extended-object approach.* In this approach one considers an extended charged object of finite size ϵ, and sum all the mutual electromagnetic forces that its various charge elements exert on each other. Then one applies the limit $\epsilon \to 0$, to obtain the self force in the point-particle limit . Obviously, if the charged object is static, the mutual forces will always cancel each other. However, if the charged object accelerates (under the influence of some external force), one generically finds that the sum of all mutual forces does not vanish. One would naturally be tempted to identify this nonvanishing "total force" as the self force acting on the particle. There is a problem, though: The resultant expression obtained for the "total force" usually includes a term that diverges like $1/\epsilon$. This divergent term must somehow be eliminated in order to obtain a physically meaningful notion of self-force. One would be tempted to apply the mass-renormalization procedure for this goal. In this procedure one re-defines the particle's rest mass so as to include the electrostatic energy $E_{es} \propto \epsilon^{-1}$. This effectively adds a term $E_{es}a^{\mu}$ to the total force, where a^{μ} is the particle's four-acceleration (we use $c = 1$ and signature $(-+++)$ throughout). Unfortunately the $O(1/\epsilon)$ term obtained in previous analyses was found to depend on the object's shape [5], and generally it does not have the form $-E_{es}a^{\mu}$ that would allow its elimination by mass-renormalization. In the special case of a charged spherical shell, Lorentz obtained an overall mutual force that diverges like $-(4/3)E_{es}a^{\mu}$, which is 4/3 times larger than what required. Several authors later confirmed the presence of this problematic 4/3 factor in spherically-symmetric charge distributions [5–8].* For non-spherical configurations the situation was found to be even worse [5]: in this case the divergent $O(1/\epsilon)$ term is not even co-directed with a^{μ}, thereby rendering the mass-renormalization procedure totally inapplicable.

In this manuscript we show that the problem described above stemmed from applying a too naive notion of "total force" (namely, a too naive summation scheme for the mutual forces). Based on energy-momentum conservation and proper relativistic kinematics, we formulate the correct method of summation. By applying this summation method to the mutual forces we obtain a universal (i.e. shape-independent) $O(1/\epsilon)$ term, which is precisely of the form $-E_{es}a^{\mu}$, so it is fully annihilated by mass-renormalization. This resolves the 4/3 problem, as well as the more general, more severe (but less well known) directionality problem associated with the $O(1/\epsilon)$ term in non-spherical configurations. Then from the remaining $O(\epsilon^0)$ term we obtain a universal expression for the self force (as $\epsilon \to 0$), which coincides with Dirac's [4] well known formula. The extended-object approach thus provides a simple and consistent interpretation of the self force in terms of standard, non-singular, classical electrodynamics.

We first study an elementary configuration of a "dumbbell" [5] which consists of two point charges situated at the edges of a short rod of fixed length 2ϵ. This analysis is then naturally generalized to include any charge distribution. The two electric charges are denoted q_+

*One should not confuse this with another "4/3 problem", namely the ratio between the momentum and energy of the electromagnetic field of a slowly-moving charged particle. These are two distinct (though perhaps related) problems; see [5]. Poincare [9] proposed a solution to the second problem, which involves the non-electromagnetic internal stresses supporting the charged object. These short-range internal stresses do not contribute, however, to the overall mutual force (they merely contribute to the bare mass).

and q_- (the subscripts "+"and "-" are used throughout to denote quantities associated with the two dumbbell's edges).

Consider first the dumbbell kinematics. The dumbbell's motion is represented by its central point, whose proper time and worldline are denoted by τ and $z^\mu(\tau)$, respectively. The four-velocity and four-acceleration of the central worldline are defined by $u^\mu \equiv \dot{z}^\mu$ and $a^\mu \equiv \dot{u}^\mu$, respectively, where an overdot denotes differentiation with respect to τ. (We allow here an arbitrary acceleration $a^\mu(\tau)$, presumably caused by an arbitrary external force f^μ_{ext} acting on the dumbbell.) At any given moment τ the rod's edges are located at spacetime's points $z^\mu_\pm(\tau)$, given by

$$z^\mu_\pm(\tau) = z^\mu(\tau) \pm \epsilon w^\mu(\tau) \ , \qquad (1)$$

where $w^\mu(\tau)$ is a unit spatial non-rotating vector (namely, w^μ satisfies $w_\mu w^\mu = 1$, $w^\mu u_\mu = 0$, as well as the Fermi-Walker transport equation, see e.g. [10]).

We denote the proper times along the worldlines of the dumbbell's two edges by $\tau_\pm$. The corresponding four-velocities and four-accelerations are $u^\mu_\pm \equiv dz^\mu_\pm/d\tau_\pm$ and $a^\mu_\pm = du^\mu_\pm/d\tau_\pm$, respectively. A straightforward calculation yields [11]

$$u^\mu_\pm = u^\mu \ , \ \frac{d\tau_\pm}{d\tau} = 1 \pm \epsilon a_{||} \ , \ a^\mu_\pm = \frac{a^\mu}{1 \pm \epsilon a_{||}} \ , \qquad (2)$$

where $a_{||} \equiv a_\lambda w^\lambda$ is the projection of a^μ on the rod's direction. The first of these equalities implies that, in the rest frame of the central point, the dumbbell's edges (and similarly any other dumbbell's point) are at rest, signifying this frame as the rest frame of the entire dumbbell.

Consider next the mutual electromagnetic forces between the two charges. Each charge feels a Lorentz force

$$f^\mu_\pm = q_\pm F^{\mu\nu}_\pm u_\nu \ ,$$

where $F^{\mu\nu}_\pm$ is the retarded electromagnetic-field tensor produced by the other charge $q_\mp$, evaluated at $z^\mu_\pm(\tau)$. To evaluate these forces we use a local expansion, derived by Dirac [4] for the retarded electromagnetic field near a point charge. This expansion, combined with the kinematical relations (2) yields the following expression valid up to order ϵ^0:

$$f^\mu_\pm \cong q_+ q_- \left[\pm \frac{w^\mu}{4\epsilon^2} - \frac{a^\mu + w^\mu a_{||}}{4\epsilon} + \frac{2}{3}(\dot{a}^\mu - a^2 u^\mu) \pm Z^\mu \right] . \qquad (3)$$

Here Z^μ is a certain $O(\epsilon^0)$ quantity (which is the same for both charges), whose explicit form is not required here as it always cancels out upon summation. Throughout this paper, the symbol "$\cong$" represents equality up to terms that vanish as $\epsilon \to 0$.

In the standard approach (see e.g. [5]) one simply sums the two mutual forces to obtain f^μ_{sum}:

$$\begin{aligned} f^\mu_{sum} &\equiv f^\mu_+ + f^\mu_- \\ &\cong -\frac{q_+ q_-}{2\epsilon}(a^\mu + w^\mu a_{||}) + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) \, . \end{aligned} \qquad (4)$$

The $O(\epsilon^{-1})$ term of this quantity suffers from the serious problem indicated above. The part proportional to a^μ is well understood, but the second part proportional to $w^\mu a_{||}$ is problematic. This second part is directed along the rod, not in the direction of a^μ, so it cannot be annihilated by mass-renormalization. When integrated over a spherical shell, this second part yields $-(1/3)E_{es}a^\mu$ [5], which may be recognized as the origin of the "4/3 problem".

This problematic $O(\epsilon^{-1})$ term indicates that something is wrong in the identification of f^μ_{sum} with the "overall mutual electromagnetic force". We shall now apply energy-momentum considerations to resolve this puzzle. Let us denote the total dumbbell's non-electromagnetic four-momentum, at a given moment τ, by $p^\mu(\tau)$. This momentum can be expressed as an integral of the dumbbell's stress-energy tensor (not including the electromagnetic stress-energy) over the dumbbell's momentary rest-frame. From energy-momentum conservation, p^μ can change only due to external forces acting on the dumbbell, and due to energy-momentum exchange with the electromagnetic field. Let us denote by dp^μ_{mut} the contribution of the mutual electromagnetic forces to this change in p^μ, during an infinitesimal time interval $d\tau$. Since the four-momentum is a conserved additive quantity, we may write dp^μ_{mut} as the sum of the contributions of the two charges. The contribution coming from the $\pm$ charge to dp^μ_{mut} is simply the mutual force $f^\mu_\pm$ acting on this charge, multiplied by the proper time $d\tau_\pm$ lapsed by this charge (between the two "moments" τ and $\tau + d\tau$; see figure 1) [12]. Namely,

$$dp^\mu_{mut} = f^\mu_+ d\tau_+ + f^\mu_- d\tau_- = \left[f^\mu_+ \frac{d\tau_+}{d\tau} + f^\mu_- \frac{d\tau_-}{d\tau} \right] d\tau \, . \qquad (5)$$

We can now identify the term in squared brackets as the "overall electromagnetic mutual force", which we denote f^μ_{mut}. Note that although the two quantities $d\tau_\pm/d\tau$ differ from unity (and from each other) only by an $O(\epsilon)$ quantity, they multiply the large quantities $f^\mu_\pm \propto O(\epsilon^{-2})$; hence the difference between f^μ_{mut} and f^μ_{sum} may be of order ϵ^{-1}. Indeed a straightforward calculation based on Eqs. (2,3,5) yields

$$f^\mu_{mut} \cong -E_{es} a^\mu + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) \ , \qquad (6)$$

where $E_{es} \equiv q_+ q_- / 2\epsilon$ is the dumbbell's electrostatic energy. Note that the $O(\epsilon^{-1})$ term of f^μ_{mut} has precisely the right form so as to be cured by mass renormalization, as we shortly describe.

The remaining $O(\epsilon^0)$ term appears somewhat problematic at first glance: It is proportional to the product

q_+q_-, whereas from basic considerations the self force (which one would like to obtain from f^μ_{mut}, after mass-renormalization, at the limit $\epsilon \to 0$) should be proportional to the square of the total charge $q \equiv q_+ + q_-$. This apparent inconsistency is resolved by noting that the mutual-forces contribution $f^\mu_{mut} d\tau$ is *not* the entire momentum exchange with the electromagnetic field: In addition to the mutual forces $f^\mu_\pm$, each charge $q_\pm$ also feels its own *self force*, which we denote $\hat{f}^\mu_\pm$. We shall refer to $\hat{f}^\mu_\pm$ as the *partial self forces* (to distinguish them from the overall self force acting on the dumbbell). Obviously it would be inconsistent to ignore the partial self forces, since our analysis yields a non-vanishing overall electromagnetic force at the limit $\epsilon \to 0$ (and by universality considerations, this result should also apply to the individual charges $q_\pm$). The overall electromagnetic force acting on the dumbbell, to which we shall refer as the "bare self force" f^μ_{bare}, is thus the sum of f^μ_{mut} and the partial self forces:

$$f^\mu_{bare} \cong \left[-E_{es} a^\mu + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) \right] + (\hat{f}^\mu_+ + \hat{f}^\mu_-) \, .$$

(Although the two new quantities $\hat{f}^\mu_\pm$ are apriori unknown, later we shall use a simple argument to relate them to the overall dumbbell self force, which will allow us to factor them out.)

We now implement the mass-renormalization procedure: We start from the dumbbell's "bare" equation of motion $m_{bare} a^\mu = f^\mu$, where $f^\mu = f^\mu_{ext} + f^\mu_{bare}$ is the total ("bare") force acting on the dumbbell, and m_{bare} represents the dumbbell's "bare mass", i.e. the total dumbbell's non-electromagnetic energy (in the momentary rest frame). We define the "renormalized mass" $m_{ren} \equiv m_{bare} + E_{es}$. The equation of motion now takes the form $m_{ren} a^\mu = f^\mu_{self} + f^\mu_{ext}$, where

$$f^\mu_{self} \equiv f^\mu_{bare} + E_{es} a^\mu$$

is the "renormalized self force". Note that f^μ_{self} has no $O(\epsilon^{-1})$ term, so we can now safely take the limit $\epsilon \to 0$ (after which the approximate equality becomes a precise one). We find

$$f^\mu_{self} = \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) + (\hat{f}^\mu_+ + \hat{f}^\mu_-) \, . \tag{7}$$

Consider next the relation between f^μ_{self} and $\hat{f}^\mu_\pm$. Since the self force is the force that a charge experiences due to its own field, it must scale (for a prescribed worldline) like the square of the particle's charge. Therefore, the above three self-forces must be related by $\hat{f}^\mu_\pm = (q^2_\pm / q^2) f^\mu_{self}$. Subtracting $\hat{f}^\mu_+ + \hat{f}^\mu_-$ from both sides of Eq. (7), and noting that

$$f^\mu_{self} - (\hat{f}^\mu_+ + \hat{f}^\mu_-) = (2 q_+ q_- / q^2) f^\mu_{self} \, ,$$

we finally obtain the desired expression for the self force:

$$f^\mu_{self} = \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu) \, . \tag{8}$$

This agrees with Dirac's expression [4], and unlike the $O(\epsilon^0)$ term in Eq. (6) it is independent of the dumbbell's charge distribution (it only depends on the total charge q).

The various elements of the above construction of f^μ_{self} can be summarized by a single mathematical expression:

$$f^\mu_{self} = \frac{q^2}{2 q_+ q_-} \lim_{\epsilon \to 0} \left[(1 + \epsilon a_{||}) f^\mu_+ + (1 - \epsilon a_{||}) f^\mu_- + \frac{q_+ q_-}{2\epsilon} a^\mu \right] \, , \tag{9}$$

whose all elements have clear meaning and justification, as discussed above (recall $d\tau_\pm / d\tau = 1 \pm \epsilon a_{||}$).

The above analysis can easily be generalized to include a general charge distribution: either an extended object consisting of N point charges, or a continuous charge distribution. (The full analysis will be given elsewhere [11]). Essentially one needs to sum over the contributions of each pair of charges (or charged volume elements) to the overall mutual force; and for each such pair, the contribution is given by the above dumbbell-model analysis. In the continuous case there is no need to consider the "partial self forces" as their contribution vanishes. (This can easily be seen from the limit $N \to \infty$ of the discrete model, in which the individual charges scale like $1/N$, and correspondingly the partial self forces scale like $1/N^2$.) In both the discrete and continuous cases, we obtain the result (8), with q being the total charge.

We conclude that at the limit $\epsilon \to 0$ the "total electromagnetic force" acting on any extended charged object is *universal*, which provides a simple interpretation to the notion of self force.

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[8] We should also mention here Fermi's work on the extended-object self force: E. Fermi, Nuovo Cimento **25**, 159 (1923). However, Fermi derived the self-force by constructing an action for the extended object, and not by summing the mutual forces. Moreover, Fermi's derivation only concerned the case of spherical symmetry and uniform acceleration.
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[10] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973), §6.5.
[11] A. Ori and E. Rosenthal, in preparation.
[12] After this work was completed, we became aware of an apparently related approach that was applied by Pearle to the case of spherical shell, in order to overcome the "4/3 problem" [13]. Our analysis, however, is simpler, more transparent, and much more general, as it applies to any charge distribution. In particular, it solves the more severe directionality problem in the non-spherical case.
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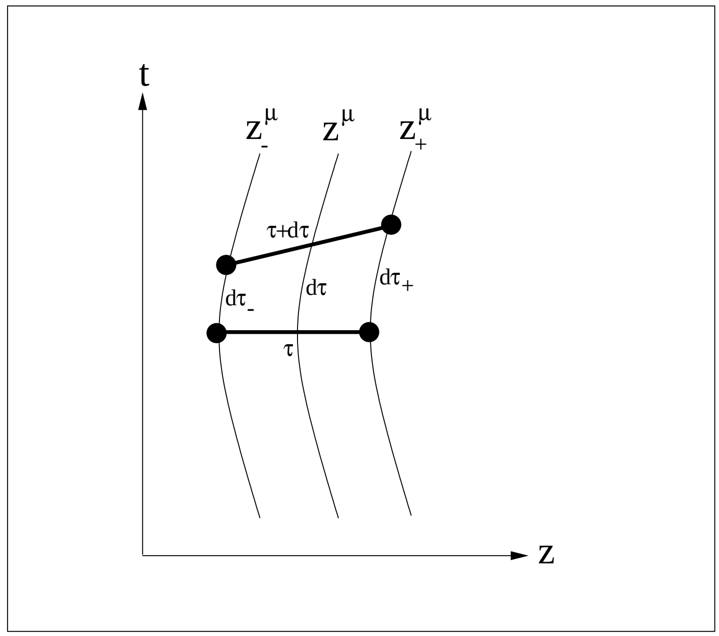

FIG. 1. A spacetime diagram describing the dumbbell's kinematics. t is the time coordinate (in some inertial reference frame), and z schematically represents a spatial coordinate. The dumbbell is represented by a straight bold line, with the black dots representing the two edge points $z^\mu_\pm$. Two such bold lines are shown, representing the dumbbell's location in spacetime at two moments separated by an infinitesimal time interval $d\tau$. The three thin solid lines are the worldlines of the central point z^μ and the two edge points $z^\mu_\pm$.